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CONNECTIVITY ANALYSIS AND APPLICATIONS OF GRAPHIC FUZZY MATROIDS

O. K. Shabna and K. Sameena

Department of Mathematics,
MES Mampad College(Autonomous),
Malappuram, Kerala - 676542, INDIA

E-mail : shabnaok@mesmampadcollege.edu.in,
sameena@mesmampadcollege.edu.in

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Abstract: This paper addresses a basic inquiry into the connectivity of fuzzy matroids, a fundamental concept with wide-ranging applications. Our contribution involves introducing an innovative equivalence relation derived from graphic fuzzy matroids. We rigorously define the connectivity of graphic fuzzy matroids using equivalence classes, providing a clear and precise characterization of their connectedness. Throughout the analysis, we highlight several essential properties of this connectivity concept, supplementing our discussion with illuminating examples. We present three practical applications showcasing the significance of connected graphic fuzzy matroids in diverse fields. In social network analysis, our findings offer valuable insights into the structure and connectivity of complex networks. In environmental reliability monitoring, connected graphic fuzzy matroids serve as a powerful tool for assessing and ensuring the reliability of environmental systems. In the context of geographic information systems, our research contributes to the enhancement of spatial connectivity analysis.

Keywords and Phrases: Graphic fuzzy matroids, Connectedness, Fuzzy Graphs.

2020 Mathematics Subject Classification: 05C40, 05C72, 05C99.

1. Introduction

The exploration of connectivity within the realm of fuzzy matroids stands as a pivotal endeavor, given the fundamental role that connectivity plays in the practical applications of these mathematical structures. Connectivity, a cornerstone concept, not only enriches the theoretical understanding of fuzzy matroids but also holds significant implications for their real-world utility. Building upon our prior investigations into graphic fuzzy matroids, this paper extends our inquiry to delve into the nuanced aspects of connectivity within this specific class of fuzzy matroids.

In the pursuit of understanding connectivity in graphic fuzzy matroids, we introduce an essential equivalence relation, establishing a framework that facilitates the systematic analysis of connectivity. The utilization of equivalence classes serves as a key tool in our exploration, allowing for a nuanced examination of the interrelationships within graphic fuzzy matroids.

Our investigation introduces a pioneering approach to analyzing connectivity in graphic fuzzy matroids, leveraging an equivalence relation derived from these structures. This novel method not only extends the theoretical foundations of fuzzy matroid theory but also provides a fresh perspective on the interplay between graph theory and matroid theory. This innovation equips us to navigate the intricacies of graphic fuzzy matroids more effectively, promising a more nuanced understanding of their connectivity properties.

Furthermore, the significance of our study extends beyond theoretical advancements. The practical relevance of connected graphic fuzzy matroids unfolds in three distinct applications such as this study provides a powerful tool for unraveling connectivity patterns in social networks, offering insights into relationship dynamics and network structures within fuzzy environments, facilitates a nuanced understanding of connectivity, aiding in the design of robust monitoring systems for environmental processes and the application of connected graphic fuzzy matroids in GIS enhances spatial analysis, enabling a more sophisticated examination of connectivity in geographic datasets with fuzzy attributes.

The motivation for this study stems from the increasing need for a comprehensive understanding of connectivity within the nuanced framework of graphic fuzzy matroids. As these structures find applications in diverse fields, ranging from network design to combinatorial optimization, a robust exploration of connectivity becomes indispensable. However, the current body of research lacks a systematic and comprehensive analysis of connectivity tailored specifically to graphic fuzzy matroids. This lacuna hampers the broader applicability and effectiveness of existing methodologies in addressing the unique challenges posed by fuzzy uncertainty in these structures.

This paper seeks to fill this critical gap by presenting a rigorous investigation into the connectivity of graphic fuzzy matroids. Our primary objective is to provide a methodological framework that not only addresses the limitations of current approaches but also enhances the theoretical and practical understanding of connectivity within this specific class of fuzzy matroids.

The existing body of research, while rich in insights, falls short in its tailored treatment of graphic fuzzy matroids. Current methodologies often overlook the intricate interplay between fuzzy set theory and matroid structures, limiting their efficacy in capturing the nuanced connectivity properties of graphic fuzzy matroids.

By elucidating the motivation, problem statement, objective, gap in existing research, and the novelty of our conducted research, we aim to set the stage for a comprehensive exploration into the connectivity of graphic fuzzy matroids in the subsequent sections of this paper.

2. Preliminaries

The concept of matroids was first proposed by Whitney in 1935 as a generalization of both graphs and linear independence in vector spaces. Matroid theory can be applied in some combinatorial optimization problems as an abstract generalization of a graph and a matrix. Graphic matroids form a fundamental class of matroids, there has been a focus of active research during the last few decades. Matroids were generalized to fuzzy fields by Goetschel and Voxman [5] using the notion of fuzzy independent set.Their works on the fuzzification of matroids preserves many basic properties of (crisp) matroids. From then on, fuzzy bases, fuzzy circuits, fuzzy rank functions and fuzzy closure operators are widely studied [[5] - [6]].

Definition 1. [8] *A Matroid M is an ordered pair (E, I) consisting of a finite set E and a collection I of subsets of E satisfying the following three conditions:*

i. $\phi \in I$

ii. *If* $A \in I$ *and* $A^{'} \subseteq A$, *then* $A^{'} \in I$

iii. *If* A_1 *and* A_2 *are in* I *and* $|A_1| < |A_2|$, *then there is an element* e *of* $A_2 - A_1$ *such that* $A_1 \cup e \in I$

The members of I are the independent sets of M, the subsets of E which are not members of I are the dependent sets of M and E is the ground set of M.

Definition 2. [8] *Let E_1 and E_2 be two finite sets. Suppose that $M_1 = (E_1, I_1)$ and $M_2 = (E_2, I_2)$ are two matroids. M_1 and M_2 are isomorphic if there exists a mapping $\psi : E_1 \longrightarrow E_2$ such that ψ satisfies the following conditions:*

i. ψ *is a one-to-one correspondence,*

ii. For each $X \subseteq E_1, X \in I_1$ *if and only if* $\psi(X) \in I_2$*,*

denoted by $M_1 \cong M_2$*. The mapping* ψ *s called an isomorphic mapping from* M_1 *to* M_2*.*

The matroid derived from a graph G is called the *cycle matroid* or *polygon matroid* of G, denoted by $M(G)$. Clearly a set X of edges is independent in $M(G)$ if and only if X does not contain the edge set of a cycle.
A matroid that is isomorphic to the cycle matroid of a graph is called *graphic matroid.*

Geotschel and Voxman generated the concept of fuzzy matroids as follows.

Definition 3. [5] *Suppose that* E *is a finite set and that* $\mathscr{I} \subseteq \mathscr{F}(E)$ *is a nonempty family of fuzzy sets satisfying:*

i. (Hereditary property) If $\mu(x) \in \mathscr{I}, \nu \in \mathscr{F}(E)$*, and* $\nu < \mu$*, then* $\nu \in \mathscr{I}$

ii. (Exchange property) If $\mu, \nu \in \mathscr{I}$ *and* $|supp\ \mu| < |supp\ \nu|$*, then there exists* $\omega \in \mathscr{I}$ *such that*

a. $\mu < \omega < \mu \vee \nu$

b. $m(\omega) \geq min\{m(\mu), m(\nu)\}$*.*

Then the pair $\mathfrak{M} = (E, \mathscr{I})$ *is a fuzzy matroid on* E*, and* $\mathscr{I}$ *is the family of independent fuzzy sets of* μ *.*

Theorem 1. [5] *Let* $\mathfrak{M} = (E, \mathscr{I})$ *be a fuzzy matroid, and for each* r*,*
$0 < r \leqslant 1$*, let*

$$I_r = \{C_r(\mu) \mid \mu \in \mathscr{I}\}.$$

Then for each $r, 0 < r \leqslant 1$*,* $M_r = (E, I_r)$ *is a (crisp) matroid on* E*.*

O.K Shabna and K. Sameena invented the concept of graphic fuzzy matroids in [17] as follows.

Theorem 2. [17] *Let* $G = (V, \sigma, \mu)$ *be a fuzzy graph with corresponding graph* $G^* = (V, E)$*. For each* r*,* $0 < r \leq 1$*, let*

$$E_r = \{e \in E \mid \mu(e) \geq r\}$$
$$\mathscr{F}_r = \{F \mid F\ is\ a\ forest\ in\ the\ (crisp) graph\ (V, E_r)\}$$
$$\mathscr{E}_r = \{\mathcal{E}(F) \mid F\ \in \mathscr{F}_r\}, where\ \mathcal{E}(F)\ is\ the\ edge\ set\ of\ F.$$

If

$$\mathscr{I} = \{\mu \in \mathscr{F}(E) \mid C_r(\mu) \in \mathscr{E}_r \; for \; each \; r, \; 0 < r \leq 1 \}$$

then, $(E, \mathscr{I})$ is a fuzzy matroid and is named as graphic fuzzy matroid.

Muhammad Akram and Fariha Zafar generalized the concept of connectivity of fuzzy graphs to rough fuzzy digraphs (RFDs) [13].

Definition 4. [13] *Let U be a universe and $\mathcal{R}$ an equivalence relation on U. Let $X \in F(U)$, where $F(U)$ represents the fuzzy power set. The lower and upper approximations of the fuzzy set X, represented by $\mathcal{R}X$ and $\overline{\mathcal{R}}X$, respectively, are characterized by fuzzy sets in U such that, for all $x_0 \in U$,*

$$\begin{aligned}(\underline{R}X)(x_0) &= \bigwedge_{z_0 \in U} \left((1 - R(x_0, z_0)) \vee X(z_0)\right), \\ \left(\overline{R}X\right)(x_0) &= \bigvee_{z_0 \in U} \left(R(x_0, z_0) \wedge X(z_0)\right)\end{aligned}$$

The pair $RX = (\underline{R}X, \overline{R}X)$ is called a rough fuzzy set.

Definition 5. [13] *A rough fuzzy digraph on a nonempty set X^* is an 4-ordered tuple $\mathcal{G} = (\mathcal{R}, \mathcal{R}X, \mathcal{S}, \mathcal{S}Y)$ such that*

i. R is s an equivalence relation on X^*,

ii. S is an equivalence relation on $Y^* \subseteq X^* \times X^*$,

iii. $RX = (\underline{R}X, \overline{R}X$ is a rough fuzzy set on X^*,

iv. $SY = (\underline{S}Y, \overline{S}Y$ is a rough fuzzy set on Y^*,

v. (RX, SY) is a fuzzy digraph, where

$\underline{\mathcal{G}} = (\underline{R}X, \underline{S}Y)$ and $\overline{\mathcal{G}} = (\overline{R}X, \overline{S}Y)$ are lower and upper approximate fuzzy digraphs of $\mathcal{G}$ such that

$$\begin{aligned}(\underline{S}Y)(z_0 z_1) &\leq min\{(\underline{R}X)(z_0), (\underline{R}X(z_1))\}, \\ (\overline{S}Y)(z_0 z_1) &\leq min\{(\overline{R}X)(z_0), (\overline{R}X(z_1))\}, \\ &\forall z_0, z_1 \in X^*\end{aligned}$$

Definition 6. [13] *The strength of connectedness between vertices z_0 and z_1 in a Fuzzy digraph $\underline{\mathcal{G}}$ is defined as the strength of connectedness from z_0 to z_1 and strength of connectedness from z_1 to z_0 and denoted by $CONN_{\underline{\mathcal{G}}}(z_0, z_1)$ and*

$CONN_{\underline{\mathcal{G}}}(z_1, z0)$, respectively. $CONN_{\underline{\mathcal{G}}}(z_0, z_1)$ is equal to the maximum of strengths of all the paths from z_0 to z_1. Also, $CONN_{\underline{\mathcal{G}}}(z_0, z_1) \neq CONN_{\underline{\mathcal{G}}}(z_1, z0)$.

Muhammad Akram Et al. discussed the connectivity parameters of m-polar fuzzy graphs in [15].

Definition 7. [15] *Let $G = (\mu, \sigma)$ be a non-trivial connected m-Polar Fuzzy graph. The m-polar fuzzy vertex connectivity of G is denoted by $\kappa(G) = (P_1\kappa(G), P_2 \kappa(G), ..., P_m\kappa(G))$ in which each j-th $(1 \leq j \leq m)$ is defined as $P_j\kappa(G) = \min P_jS_W(\overline{X})$; $i \in \omega$, i.e., it is the minimum of strong weights of m-PF vertex cuts of G.*

In [14], Muhammed Akram Et al. introduced the connectivity and average connectivity indices of m-polar fuzzy graphs.

Definition 8. [14] *Let $G = (\zeta, \sigma)$ be an m-Polar Fuzzy graph. The connectivity index of G is denoted by $CI(G)$ and is defined as $CI(G) = (P_1 \circ CI(G), P_2 \circ CI(G), \cdots, P_m \circ CI(G))$, where $P_q \circ CI(G) = \sum_{w,z \in W} P_q \circ \zeta(w) P_q \circ \zeta(z) P_q \circ CONN_G(w, z)$ for each $1 \leq q \leq m$.*

Proposition 1. [14] *Let $G = (\zeta, \sigma)$ be an m-Polar Fuzzy graph and $\overline{G} = \overline{\zeta}, \overline{\sigma}$ and be an m-PF subgraph of G. Then for each $1 \leq q \leq m, P_q \circ CI(\overline{G}) \leq P_q \circ CI(G)$, that is, $CI(\overline{G})$ will be less than or equal to $CI(G)$.*

Definition 9. [14] *Let $G = (\zeta, \sigma)$ be an m-Polar Fuzzy graph. The average connectivity index of G is denoted by $ACI(G)$ and is defned as $ACI(G) = (P_1 \circ ACI(G)), P_2 \circ ACI(G), \cdots, (P_m \circ ACI(G))$, where $P_1 \circ ACI(G) = \frac{1}{{}^nC_2} \sum_{w,z \in W} P_1 \circ \zeta(w) P_1 \circ \zeta(z) P_1 \circ CONN_G(w, z)$ for each $1 \leq q \leq m$.*

3. Connectivity for Graphic Fuzzy Matroids

In this part, our focus now narrows down to the captivating domain of connectivity within graphic fuzzy matroids. The inherent uncertainties characterizing real-world networks are summarized through the lens of graphic fuzzy matroids, where the strength of connections becomes a key aspect of analysis. We begin by recalling the definition of connectedness for fuzzy graphs.

Let $G = (V, \sigma)$ be a fuzzy graph, let x, y be two distinct vertices and let G' be the partial fuzzy subgraph of G obtained by deleting the edge xy. That is, $G' = (V, \sigma')$, where $\mu'(xy) = 0$ and $\mu' = \mu$ for all other pairs. We call xy a **fuzzy bridge** in G if $\mu'^{\infty}(u, v) < \mu^{\infty}(u, v)$ for some u, v in σ^*. Let w be any vertex and w is a **fuzzy cutvertex** if deleting the vertex w reduces the strength of connectedness between some other pair of vertices. Hence, w is a fuzzy cutvertex if and only if there exists u, v distinct from w such that w is on every strongest path from u to v. A partial fuzzy subgraph G' of G is called **nonseparable** or a **block** if it has no fuzzy cutvertices. Sometimes we refer to a block in a fuzzy graph as a fuzzy block.

A **maximum spanning tree** of a connected fuzzy graph (σ, μ) is a fuzzy spanning subgraph $T = (\sigma, \nu)$ of G, which is a tree, such that $\mu^{\infty}(u, v)$ is the strength of the unique strongest $u - v$ path in T for all $u, v \in G$.

Theorem 3. [9] *Let $\mathfrak{M} = (E, \mathscr{I})$ be a fuzzy matroid, $e_1, e_2, e_3 \in [0,1]^E$ and $\mu, \nu \in \mathscr{C}(\mathscr{M})$. If $e_1 \vee e_2 \leq \mu, e_2 \vee e_3 \leq \nu$ and $\mathscr{C}(\mu) \cap \mathscr{C}(\nu) \neq \phi$, then there is a fuzzy circuit ω such that $e_1 \vee e_3 \leq \omega$.*

We now define a kind of connectedness in graphic fuzzy matroids and present some useful properties of them.

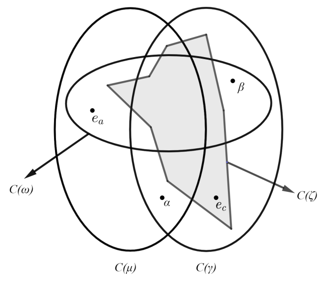
Let $\mathfrak{M}$ be a graphic fuzzy matroid. Define a relation $\sim$ on the fuzzy edge set μ of the fuzzy graph $G = (V, \sigma, \mu)$ in such a way that, for any two fuzzy edges $e_a, e_b \in \mu$, $e_a \sim e_b$ if and only if either there is a fuzzy circuit $\mathcal{C}$ in $\mathfrak{C}(\mathfrak{M})$ containing both e_a and e_b or $e_a = e_b$.

Proposition 2. *The relation $\sim$ is an equivalence relation on σ.*

Proof. Clearly $\sim$ is reflexive and symmetric. To show that $\sim$ is transitive, suppose that $e_a,\ e_b,\ e_c \in [0,1]^E$ and $\mu,\ \nu \in \mathfrak{C}(\mathfrak{M})$ such that $e_a \vee e_b \leq \mu$ and $e_b \vee e_c \leq \nu$. Also, $\mathcal{C}(\mu) \cap \mathcal{C}(\nu) \neq \phi$. Corresponding to the fuzzy circuits $\mathcal{C}(\mu)$ and $\mathcal{C}(\nu)$, there is circuits C_1 and C_2, respectively, in the crisp matroid such that $C_1 \cap C_2 \neq \phi$. Assume that, $C_1 \cup C_2$ is the minimal circuit in the collection of all such pairs of circuits.

Claim: There is a fuzzy circuit $\mathcal{C}$ such that $e_a \vee e_c \leq \mathcal{C}$.

Assume that there is no such a fuzzy circuit. Then obviously $\mathcal{C}(\mu) \neq \mathcal{C}(\nu)$ and hence $C_1 \neq C_2$. Choose an element $\alpha \in \mathcal{C}(\mu) \cap \mathcal{C}(\nu)$. In the following Figure, given a Venn diagram useful in keeping track of the rest of the proof.



By Theorem 3, $\mathfrak{M}$ has a fuzzy circuit, let it be $\mathcal{C}(\omega)$, such that $e_a \in \mathcal{C}(\omega) \leq (\mathcal{C}(\mu) \wedge \mathcal{C}(\nu)) \setminus \backslash\alpha$. It is clear that, by assumption, $e_c \notin \mathcal{C}(\omega)$. As $\mathcal{C}(\omega) \nleq \mathcal{C}(\mu)$, there exists an element β of $\mathcal{C}(\nu) \backslash \mathcal{C}(\mu)$ that is in $\mathcal{C}(\omega)$. Applying Theorem 3 to $\mathcal{C}(\nu)$ and $\mathcal{C}(\omega)$, we find that $\mathfrak{M}$ has a fuzzy circuit $\mathcal{C}_\zeta$ such that $e_c \in \mathcal{C}_\zeta < (\mathcal{C}(\nu) \vee \mathcal{C}(\omega)) - \beta$. Since $\mathcal{C}_\zeta \nless \mathcal{C}(\nu)$, the fuzzy set $C_\zeta \wedge (\mathcal{C}(\omega) - \mathcal{C}(\nu)) \neq \phi$. Therefore $\mathcal{C}_\zeta \wedge \mathcal{C}(\mu) \neq \phi$.

But $\mathcal{C}(\mu) \vee \mathcal{C}_\zeta \leq (\mathcal{C}(\mu) \vee \mathcal{C}(\nu)) - \beta$ and so $\mid \mathcal{C}_1 \bigcup \mathcal{C}_3 \mid < \mid \mathcal{C}_1 \bigcup \mathcal{C}_2 \mid$, where $\mathcal{C}_3$ is the circuit in the crisp matroid corresponding to the fuzzy circuit $\mathcal{C}(\zeta)$. Therefore the pair $(\mathcal{C}_1, \mathcal{C}_3)$ contradicts the choice of $(\mathcal{C}_1, \mathcal{C}_2)$ because $e_a \in \mathcal{C}(\mu)$, $e_c \in \mathcal{C}(\zeta)$ and $\mathcal{C}(\mu) \wedge \mathcal{C}(\zeta) \neq \phi$.

If there is only one equivalence class determined by this equivalence relation, then we call $\mathfrak{M}$ a **connected graphic fuzzy matroid**. Suppose that $\nu \in [0,1]^X$. If for every pair of fuzzy points $x_a, y_b \leq \nu$, there exists $\omega \in C(\mathfrak{M})$ such that $x_a, y_b \leq \omega \leq \nu$, then ν is said to be connected in $\mathfrak{M}$. We can obtain the following as an immediate consequence of Proposition 1.

Proposition 3. *Let $\mathfrak{M} = (E, \mathscr{I})$ be a graphic fuzzy matroid, then $\mathfrak{M}$ is connected if and only if for every pair of distinct fuzzy edges e_1 and e_2 of $E(\mathfrak{M})$, there is a fuzzy circuit $C \in \mathscr{C}$ containing both e_1 and e_2.*

The following result characterizes a connected graphic fuzzy matroid in terms of connected crisp cyclic matroids.

Proposition 4. *Let $\mathfrak{M} = (E, \mathscr{I})$ be a graphic fuzzy matroid, then $\mathfrak{M}$ is connected if the underlying cyclic matroid is connected.*

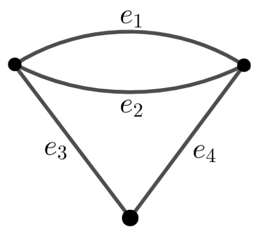
Proof. Suppose that $\mathfrak{M}$ is connected. Let $e_a, e_b \in E$ with $e_a \neq e_b$. Since $\mathfrak{M}$ is connected, there exists a fuzzy circuit $\mu \in C(\mathfrak{M})$ satisfying $e_a \vee e_b \leq \mu$. By lemma 2.4, $\mu_{[m(\mu)]}$ is a circuit in the cyclic matroid $(E, \tau_{m(\mu)})$. By definition of graphic fuzzy matroids, $\mu_{m(\mu)}$ is a circuit in the underlying cyclic matroid,let it be (E, τ). Therefore (E, τ) is connected.

Conversely suppose (E, τ) is connected cyclic matroid. Let $e_a, e_b \in [0,1]^E$. Since (E, τ) is connected, there exists a circuit C in (E, τ) containing the edges corresponding to e_a, e_b. Let $\omega = \chi_C$, then $\omega \in C(\mathfrak{M})$ and $e_a, e_b \leq \omega$, so $\mathfrak{M}$ is connected.

Example 1. Let G be the graph shown in the following figure. Then the corresponding cycle matroid is $M = (E, \tau)$, where $E = \{e_1, e_2, e_3, e_4\}$ and $\tau = 2^{\{e_1, e_3, e_4\}} \cup 2^{\{e_2, e_3, e_4\}} - \{\{e_1, e_3, e_4\}, \{e_2, e_3, e_4\}\}$. Let

$$\tau_a = \begin{cases} 2^E, & \text{if } 0 < a \leq \dfrac{1}{4} \\ 2^{\{e_1,e_3,e_4\}} \cup 2^{\{e_2,e_3,e_4\}}, & \text{if } \dfrac{1}{4} < a \leq \dfrac{3}{4} \\ \tau, & \dfrac{3}{4} < a \leq 1 \end{cases} \tag{1}$$

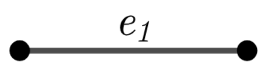
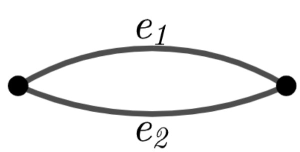
and $\mathscr{I} = \{\mu \in [0,1]^E \mid \mu_{[a]} \in \tau_a, 0 < a \leq 1\}$, then $\mathfrak{M} = (E, \mathscr{I})$ is a graphic fuzzy matroid.

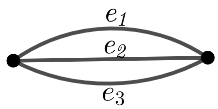
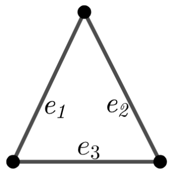
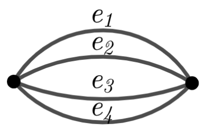
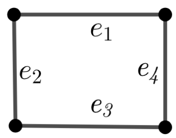


G

Note that G in the figure is 2-connected, so the corresponding cycle matroid M is connected. By proposition 2, $\mathfrak{M}$ is connected.

Example 2. The table below enumerates various connected graphic fuzzy matroids involving up to four fuzzy edges. In each instance, the fuzzy edges are assigned a value of 1. In these specified fuzzy matroids, the independence of a set of edges aligns with the concept of linearly independent spanning trees within the underlying graph.

Number of fuzzy edges	underlying fuzzy graph	Connected fuzzy matroid
0		$\mathfrak{M} = (E, \mathscr{I})$ $E = \{\,\}$ $\mathscr{I} = \{\,\}$
1		$\mathfrak{M}_1 = (E, \mathscr{I}_1)$ $E = \{e_1\}$ $\mathscr{I}_1 = \{\,\}$
		$\mathfrak{M}_2 = (E, \mathscr{I}_2)$ $E = \{e_1\}$ $\mathscr{I}_2 = \{\{e_1\}, \{\,\}\}$
2		$\mathfrak{M}_1 = (E, \mathscr{I}_1)$ $E = \{e_1, e_2\}$ $\mathscr{I}_1 = \{\{e_1\}, \{\,\}\}$

Number of fuzzy edges	underlying fuzzy graph	Connected fuzzy matroid
3		$\mathfrak{M}_1 = (E, \mathscr{I}_1)$ $E = \{e_1, e_2, e_3\}$ $\mathscr{I}_1 = \{\{e_1\}, \phi\}$
		$\mathfrak{M}_2 = (E, \mathscr{I}_2)$ $E = \{e_1, e_2, e_3\}$ $\mathscr{I}_2 = \{\phi, \{e_1\}, \{e_2\}, \{e_2\}, \{e_1, e_2\}\}$
4		$\mathfrak{M}_1 = (E, \mathscr{I}_1)$ $E = \{e_1, e_2, e_3, e_4\}$ $\mathscr{I}_1 = \{\{e_1\}, \phi\}$
		$\mathfrak{M}_1 = (E, \mathscr{I}_2)$ $E = \{e_1, e_2, e_3, e_4\}$ $\mathscr{I}_2 = [0,1]^E - \{e_1, e_2, e_3, e_4\}$

4. Applications

4.1. Social Network Analysis

Imagine a social network where individuals are represented as nodes, and edges correspond to friendships between individuals. However, the strength or quality of these friendships is uncertain and is modeled using fuzzy values between 0 and 1. Here's how the fuzzy social interaction matroid could be defined:

Fuzzy Friendship Graph: Nodes represent individuals in a community, and edges represent friendships.Each edge is associated with a fuzzy value indicating the strength or closeness of the friendship. For example, a fuzzy value of 0.8 might indicate a strong friendship, while 0.3 might represent a weaker connection.

Independence in the Fuzzy Friendship Matroid: A set of friendships is considered independent if there are no cycles in the subgraph formed by those friendships, and the sum of the fuzzy values along any cycle is less than or equal to 1.

Connectivity Aspect: The connected graphic fuzzy matroid explores how uncertainties in the strengths of friendships influence the overall connectedness of the social network. Higher fuzzy values contribute to a more connected network, while lower fuzzy values may result in isolated clusters.

Dynamic Changes: The fuzzy values associated with edges can change over time, representing the dynamic nature of friendships in the social network. The matroid captures how these changes influence the evolving structure and connectivity of the social interaction network.

4.2. Environmental Reliability Monitoring

In the context of environmental monitoring, envision a sensor network designed to assess various environmental parameters. Each sensor, strategically placed in the field, measures specific aspects such as temperature, humidity, and pollution levels. However, the reliability of these measurements is subject to uncertainty, modeled by fuzzy values ranging from 0 to 1.

Fuzzy Graph Representation: Nodes represent individual sensors deployed for environmental monitoring. Edges symbolize communication links facilitating data exchange between neighboring sensors. Fuzzy values are assigned to each sensor reading, reflecting the reliability or confidence in the measurement.

Independence Criteria: A set of sensors is deemed independent if their measurements align within a defined tolerance based on fuzzy compatibility. The fuzzy matroid captures the intricacies of independence, considering uncertainties in sensor readings.

Connectivity Exploration: The connected fuzzy sensor network matroid investigates how uncertainties in sensor readings influence the overall connectivity of the network. Sensors with higher fuzzy reliability values contribute to a more interconnected network, whereas those with lower values may lead to isolated clusters.

Dynamic Adaptation: Fuzzy values associated with sensor readings are dynamic, capable of changing over time due to environmental fluctuations or sensor malfunctions. The matroid provides insights into how these dynamic changes impact the evolving structure and connectivity of the sensor network.

4.3. Geographic Information System

Imagine a scenario where a fuzzy geometric matroid is applied to model uncertainties in the spatial positioning of points in a geographic information system (GIS). This matroid captures the fuzzy relationships among points with uncertain coordinates.

Fuzzy Graph Formation Nodes represent geographical points in a GIS, each with fuzzy values associated with its coordinates. Fuzzy values, ranging from 0 to 1, represent the uncertainty or imprecision in the exact spatial location of each point.

Independence Criteria: A set of points is considered independent if the uncertainties in their coordinates do not lead to contradictions in the spatial relationships, given a certain tolerance level defined by the fuzzy values. The fuzzy geometric matroid captures the nuanced independence of spatial points.

Connectivity Exploration: The connected fuzzy geometric matroid investigates how uncertainties in the spatial positioning of points influence the overall connectivity of the geographic network. Points with higher fuzzy values for their coordinates contribute to a more connected spatial network, while those with lower values may lead to isolated or less precisely defined regions.

Dynamic Spatial Configurations: Fuzzy values associated with point coordinates may dynamically change over time due to factors such as environmental shifts or sensor imprecisions. The matroid provides insights into how these dynamic changes impact the evolving structure and connectivity of the fuzzy geometric network.

5. Conclusion

Connectivity, a fundamental concept in both graph theory and matroid theory, has been explored in the context of graphic fuzzy matroids in this study. We introduced the concept of connectivity to graphic fuzzy matroids, establishing a robust framework through the definition of an equivalence relation. This framework facilitated the development of connected graphic fuzzy matroids, characterized by their structure in terms of equivalence classes.

Our research has demonstrated that the connectedness of the underlying cycle matroid is not only necessary but also a sufficient condition for the connectedness of a graphic fuzzy matroid. Furthermore, we have uncovered additional properties related to connectedness, supported by illustrative examples.

We have included three practical applications of connected graphic fuzzy matroids. Firstly, in social network analysis, our method proves invaluable in understanding and modeling connectivity patterns within social structures. Secondly, in environmental reliability monitoring, the application of connected graphic fuzzy matroids offers a powerful tool for assessing and managing the reliability of environmental systems. Lastly, in geographic information systems, our approach enhances the analysis and representation of spatial connectivity.

The significance of our results extends beyond theoretical contributions. The connectivity of graphic fuzzy matroids provides profound insights with diverse practical applications, ranging from social network analysis to environmental reliability monitoring and geographic information systems. These applications underscore the versatility and real-world relevance of our proposed method.

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